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The motion generated by a body moving through a stratified fluid at large Richardson numbers

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Experiments are described in which a body was towed at large Richardson numbers through a uniformly stratified fluid, with particular attention being given to the parameter range in which diffusive effects are likely to play an important role. When the Richardson number is only moderately large (i.e. less than about 10^6) it appears that a linear viscous diffusive theory fails to model the experimental observations, but at larger Richardson numbers it seems that such a theory is a strong candidate for modelling the physical situation. At Richardson numbers of about 10^3 a non-diffusive viscous model (see the theories of Graebel 1969; Janowitz 1971) appears to give a fairly good description of the experimental results, except for some discrepancies between the theories and the observed flows in the wake downstream from the body. However, the conditions under which the experiments were carried out were not completely favourable for the application of these theories.

A brief survey and new interpretations of some of the work on the rotating-fluid counterparts of the present experiments are also given. In view of the theoretical similarities between the two situations we have tried to assess and compare (where possible) the results of experimental investigations in each field and we feel that the observations in the rotating-fluid experiments are in good qualitative agreement with the present results. No measurements of the drag acting on the body were made in the present experiments but this has been done carefully in the rotating-fluid experiments, and the results of those measurements are in poor agreement with the theoretical predictions. Because of the close similarities between the axisymmetric model for viscous rotating fluids (cf. Moore & Saffman 1969) and the two-dimensional model for diffusive stratified fluids (Freund & Meyer 1972), an explanation for the discrepancy should be sought before the linear viscous diffusive model can be applied with any confidence to the flow of stratified fluids at very large Richardson numbers.

1. Introduction

This paper describes an experimental investigation of the flows generated by a body moving horizontally at large Richardson numbers through a stratified fluid. We shall be concerned only with two-dimensional flows of fluids having a linear undisturbed density variation $\tilde{\rho}_0$ given by:†

$$\tilde{\rho}_0 = \tilde{\rho}_c(1 - \tilde{z}/h),$$

where $\tilde{z}$ is the vertical height measured from the centre of the body, h is a positive constant (usually referred to as the scale height), and $\tilde{\rho}_c$ is the density of the fluid at the level $\tilde{z} = 0$.

In describing the motions we shall refer to a rectangular Cartesian co-ordinate system whose origin is at the centre of the body and which moves with the body. Let the horizontal co-ordinate be $\tilde{x}$ and the vertical co-ordinate be $\tilde{z}$. Since we shall be interested only in the situation in which the body moves with a steady speed (relative to a reference frame fixed in the laboratory) we shall consider $\tilde{x}$ to increase in the direction in which the body moves.

The Richardson number is defined to be $Ri \equiv b^2g/(U^2h)$, where $2b$ is the height of the body and g is the acceleration due to gravity. There are, of course, a number of other parameters relating to this situation and it is necessary to qualify the term 'large Richardson number'. In particular, the important parameter range of interest here is that in which (a) the Richardson number is much larger than the Schmidt (or Prandtl) number $\sigma = \nu/\vartheta$, where ν is the kinematic viscosity and ϑ is the diffusivity of the solution, and (b) the Reynolds number $Re \equiv Ub/\nu$ is not too small. (Explicit conditions are given below in the limits (2.1).)

In recent years there have been a number of theoretical and experimental papers on the horizontal motion of bodies through stratified fluids. All of these are in agreement that, for sufficiently small speeds, there is an influence extending far ahead of the body. This very property makes it extremely difficult to interpret laboratory experiments in terms of theoretical models for which the fluid is assumed to be of infinite extent. More specifically, let us consider the conditions, for the experiments to be described below, that would be needed to provide a reasonable basis for comparisons with the theory of Freund & Meyer (1972, hereafter referred to as FM). The experiments are to be carried out in brine which we shall take to have a scale height of 600 cm, and the temperature of the fluid is taken to be 19 °C, so that its viscosity and diffusivity are known. Then, if the fluid velocity at the level $\tilde{z} = 0$ is denoted by $\tilde{u}_0$ we find from FM that $\tilde{u}_0/U = 0{\cdot}16$ at a distance from the body of $5{\cdot}7 \times 10^3 b^3$ cm and that $\tilde{u}_0/U = 0{\cdot}02$ at a distance from the body of $1{\cdot}5 \times 10^5 b^3$ cm. Thus we see that the influence of the body increases as the cube of the body height. On the other hand, for the FM theory to be applicable the height $2b$ of the body must be much larger than an intrinsic length scale $l = (h\vartheta\nu/g)^{\frac{1}{4}}$, which in the present example has the value 0·016 cm; in addition, to achieve large Richardson numbers it is convenient to choose b not too small. A good compromise appears to be the value $b \sim 0{\cdot}5$ cm, in which case the tank would need to be about 15 m long if the velocity at the position of the end walls, according to the FM theory, is to be $0{\cdot}16U$, and about 400 m long if $\tilde{u}_0/U$ is to be 0·02 at the position of the end walls. Thus, if the ends of the tank

† A tilde over a variable indicates that the variable is a physical quantity, otherwise the variable is assumed to be dimensionless.

are to have only a small influence on the motions generated by the body, it would appear that the tank should be about 100 m long. As this is usually impossible a modified experiment must be considered.

For the experiments described herein we have chosen the height $2b$ of the body to be 1 cm and the length of the tank to be 3 m, with the result that the ends should not *completely dominate* the experiment: on the basis of the FM theory the velocity at the position of the end walls, at the level $\tilde{z} = 0$, would be approximately $\tilde{u}_0 = 0{\cdot}4U$. But, since the influence of the ends was likely to be important, we decided that, rather than attempt a detailed comparison with the various theories, we should look at the gross features (such as the overall symmetry properties) of the flow and see if any of the theories could be discounted on that basis alone by our experiments.

In § 2 a brief outline is given of the theory underlying the experiments, and in § 3 a discussion is given of previous experimental work together with a table summarizing the parameter ranges covered by the experiments. We try in § 4 to discuss how the results from rotating-fluid experiments might be helpful in interpreting the results from the present study. Then, in §§ 5 and 6, the details of the present experiments are given.

2. Theory

Theoretically there is an exact analogy between two-dimensional flows of rotating fluids and two-dimensional motions of stratified fluids (cf. Veronis 1970). The fluids are assumed to be incompressible and the Boussinesq approximation must be used in the continuity and momentum equations for the stratified fluid. The analogy applies either (*a*) when both the fluids are inviscid, or (*b*) between a viscous rotating fluid and a viscous, diffusive, stratified fluid of unit Schmidt number, if the viscosities are constant. In the case of steady linear motions the Schmidt number can be incorporated in the scaling of the dependent variable and need not appear explicitly in the equations, in which case the analogy is valid for all Schmidt numbers. In addition, the theoretical solutions for linear motions of axially symmetric, rotating fluids are of a similar form to those for two-dimensional motions, and accordingly it might be expected that there would be certain similarities between axially symmetric experiments in a rotating fluid and two-dimensional experiments in a stratified fluid. The analogy for viscous two-dimensional motions applies only to diffusive stratified fluids; if the stratified fluid is not diffusive the analogy does not hold.

In this paper we are interested in investigating the motions that are realized in a stratified fluid in the limit

$$Re^{\frac{1}{2}}(\sigma Ri)^{\frac{1}{4}} \rightarrow \infty, \quad \beta \rightarrow \infty, \quad (\sigma Ri)^{\frac{1}{2}} \rightarrow \infty, \quad \sigma^{-\frac{1}{2}} Ri^{\frac{1}{2}} \rightarrow \infty, \tag{2.1}$$

where $\beta \equiv h/b$ is the Boussinesq number. In this limit the solution to the set of linear equations studied by FM is a 'limit solution' to the full equations for steady motions of a viscous, diffusive, stratified fluid. The comparison equations studied by FM can be reduced, in terms of the stream function ψ, to the single equation

$$\partial^6\psi/\partial z^6 + \partial^2\psi/\partial x^2 = 0. \tag{2.2}$$

The co-ordinates are defined in § 1: x is the horizontal co-ordinate, z is the vertical co-ordinate, and the frame of reference moves with the body. The stream function is defined such that the horizontal velocity $u = \partial\psi/\partial z$ and the vertical velocity $w = -\partial\psi/\partial x$. The boundary conditions for this equation are that a velocity condition must be satisfied on the body and that the perturbation stream function ψ should satisfy suitable growth conditions at large distances from the body. The solution to (2.2) given by FM is symmetric in x.† Thus, in particular, they find that the horizontal velocity is symmetric in x, and that the vertical velocity w and the density perturbation ρ are antisymmetric in x. Moreover, as is shown in the appendix, any solution to (2.2) satisfying symmetric boundary conditions necessarily has these symmetry properties.

For stratified fluids it is also possible to construct a theory for viscous non-diffusive fluids (see Graebel 1969; Janowitz 1971); this has no counterpart in rotating fluids. The linearized equation describing two-dimensional motions of such fluids is

$$\nabla^4\psi + \partial\psi/\partial x = 0,$$

which, in contrast to (2.2), does not admit solutions that are symmetric in x (FM discusses this in more detail). Both these non-diffusive theories yield an upstream wake extending far ahead of the body and virtually uniform flow downstream. Thus, in front of the body, there is a region of trapped (or blocked) fluid whose length L is given by

$$L/b = C(Re\,Ri), \tag{2.3}$$

where C is a constant, and beyond which the influence of the body decays gradually to zero. For a given stratification and a given size of body the length of this region of trapped fluid is inversely proportional to the speed of the body, whereas the wake to the rear of the body decays exponentially away from the body at a (spatial) rate of $\frac{1}{2}b(hg/U\nu)^{\frac{1}{4}}$. For the experiments described below this distance is about 0·02 cm. The drag per unit span on the body is infinite according to Janowitz's theory (cf. FM) and is independent of the speed of the body according to Graebel's theory.

FM point out that both Graebel's and Janowitz's non-diffusive theories are limit solutions of the exact equations of motion in the parameter limit

$$(Re\,Ri)^{\frac{1}{3}} \to \infty, \quad \beta \to \infty, \quad Ri \to \infty, \quad \sigma\beta Ri^{-1} \to \infty. \tag{2.4}$$

Theories similar to those of Bretherton (1967) and FM have been developed for the *axisymmetric* motions generated by a sphere or a disk moving slowly along the axis of a uniformly rotating fluid. The time-dependent inviscid motions generated by a sphere were determined by Stewartson (1952) and the steady

† The solution derived by FM had previously been written down by Bretherton (1967). In his paper Bretherton studied the two-dimensional initial-value problem for a circular cylinder started moving with a steady velocity along the axis of a uniformly rotating, inviscid fluid: he was able to interpret his solution completely in terms of inertial waves generated by the body. Then, for a slightly viscous fluid he argued that the waves would be generated by the body in the same manner as for the inviscid case, but that they would gradually dissipate. Using these ideas he predicted, for large times, the same flow as that calculated by FM.

viscous flow generated by a disk is described in Moore & Saffman (1969, hereafter referred to as MS). The equation for the stream function ψ considered by MS was of the form

$$\frac{\partial}{\partial y} D^4 \frac{\partial}{\partial y} \psi + y^{-1} \frac{\partial^2 \psi}{\partial x^2} = 0, \tag{2.5}$$

where $2y$ is the square of the radius and $D^2 = y\partial^2/\partial y^2 + \partial/\partial y$; the axial velocity $u = \partial\psi/\partial y$ and the radial velocity $w = -(2y)^{-\frac{1}{2}} \partial\psi/\partial x$. The boundary conditions for (2.5) are similar to those for (2.2), and the solution to (2.5) given by MS closely resembles the FM solution to (2.2). Moreover, the solution given by MS is a limit solution of the exact equations of motion in the limit (cf. FM)

$$T^{\frac{1}{2}} \rightarrow \infty, \quad N \rightarrow \infty, \tag{2.6}$$

where $T \equiv \Omega b^2/\nu$ is the Taylor number and $N \equiv 2\Omega b/U$ is the inverse of the Rossby number. Here b is the radius of the disk and Ω is the angular velocity of the fluid.

3. Previous experiments with stratified fluids

One of the first people to investigate the flow of stratified fluids at large Richardson numbers was Yih (1959); his experiments were mainly of a qualitative nature, and it is only in the past few years that detailed measurements of such flows have been made. Laws & Stevenson (1972) recently made some measurements at quite large Richardson numbers (see table 1 for details), but their experiments were carried out in a rather short tank and this makes it difficult to interpret their results in terms of the theoretical models that have been developed for infinitely long tanks. For example, some of their experiments were made with a circular cylinder of diameter 2·54 cm, for which case the FM theory indicates that $\tilde{u}_0/U$ should exceed 1·0 at distances up to 250 cm ahead of the cylinder, when the channel is infinitely long. But Laws & Stevenson made their experiments in a tank whose overall length was only 180 cm, and the measured values of $\tilde{u}_0/U$ decreased almost linearly to zero at the upstream wall of the tank. Accordingly it would appear that the ends of the tank played a dominant role in determining the motions, and indeed a theoretical model developed by Foster & Saffman (1970) for the flow of stratified fluids in short tanks contains the feature observed experimentally by Laws & Stevenson.

Browand & Winant (1972) made some measurements in a tank of moderate length,† and their experiments indicated that the ends of the tank did not play a *dominant* role in determining the motions generated by the body. These experiments showed marked asymmetries between the flow in front and to the rear of the body, and accordingly we deduce that the theory of FM cannot give a good description of the experiments. However, the parameter $Ri^{\frac{1}{2}}\sigma^{-\frac{1}{2}}$, occurring in the limit (2.1), ranged in value from 0·40 to 1·6 (cf. table 1) in these experiments, which may be insufficiently large for the FM theory to be applicable. On the other hand, the limit (2.4) for the viscous non-diffusive theories appears to have been fairly well satisfied in the experiments of Browand & Winant and accordingly such

† The ratio of the length of their tank to the height of the body lay between 190 and 1300.

FM limit (2.1)	$Re^{\frac{1}{2}}(\sigma Ri)^{\frac{1}{4}} \to \infty$	$\beta \to \infty$	$(\sigma Ri)^{\frac{1}{2}} \to \infty$	$Ri^{\frac{1}{2}}\sigma^{-\frac{1}{2}} \to \infty$
Laws & Stevenson	30–96	1500–433	$3{\cdot}1 \times 10^3$–$2{\cdot}3 \times 10^4$	3·1–23
Browand & Winant	120	3700–540	240–970	0·40–1·6
Present experiments	30	1200	$1{\cdot}6 \times 10^3$–$5{\cdot}2 \times 10^4$	1·6–67
Graebel–Janowitz limit (2.4)	$(Re\,Ri)^{\frac{1}{3}} \to \infty$	$\beta \to \infty$	$Ri \to \infty$	$\sigma\beta Ri^{-1} \to \infty$
Laws & Stevenson	14–60	1500–433	$9{\cdot}3 \times 10^{-3}$–$5{\cdot}2 \times 10^5$	161–0·83
Browand & Winant	18–28	3700–540	97–1580	3340–1400
Present experiments	11–45	1200	$2{\cdot}4 \times 10^3$–$4{\cdot}5 \times 10^6$	500–0·28

TABLE 1. A summary of the parameters involved in the limits (2.1) and (2.4)

theories are possible candidates for giving an explanation of the observed motions.

It is difficult to assess how well these theories describe Browand & Winant's observations, but the experiments revealed a well-defined wake to the rear of the body which is not predicted by either the theory of Graebel (1969) or that of Janowitz (1971).

4. A new look at the rotating-fluid experiments

Because of the similarities between the theoretical models for rotating- and for stratified-fluid flows it is helpful to consider the existing experimental results for the motions generated by bodies in rotating fluids before describing our experiments. Through different measurements and the use of different flow-visualization techniques (whichever is the most appropriate in each case) the experiments with the two flows provide different kinds of information relating to similar phenomena. The following rotating-fluid experiments appear to be the most relevant to the present study.

Drag measurements

Maxworthy (1970) has made a very careful study of the motions generated by a sphere moving along the axis of rotation of a moderately long vessel. In particular he measured the drag acting on the sphere, and for large values of T and N [cf. the limit (2.6)] he found that the drag coefficient C_D is given by

$$C_D = AN^B,$$

where $A = 2{\cdot}60 \pm 0{\cdot}05$ and $B = 1{\cdot}00 \pm 0{\cdot}01$. The theoretical values for C_D for the the viscous motions generated by a disk (cf. MS) and for the inviscid motions generated by a sphere (cf. Stewartson 1952) are the same, namely $C_D = 1{\cdot}70N$, so that the theories predict the observed dependence of C_D on N but give a value of the quantity A which differs considerably from the empirical value obtained by Maxworthy. The reason for this discrepancy is not known at present.

Maxworthy (1968) has also measured the drag on a sphere in a short container where the presence of the ends is of crucial importance. For this situation Moore & Saffman (1968) have studied a theoretical model based on free shear layers along the generators of the cylinder of radius b, and Ekman boundary layers on the body and end wall. Maxworthy's experiments were performed at values of N

as large as 10^4 and values of T up to $2{\cdot}6 \times 10^4$, and yet the measured drag differed significantly from that predicted by Moore & Saffman for large N and T.†

The column of trapped fluid ahead of the body

In his 1970 experiments Maxworthy observed "a sharply defined region of almost stagnant fluid" ahead of the sphere, and by allowing the body to emerge from a region of coloured water into a region of clear water he was able to determine the length of this slug of blocked fluid. These measurements indicated that, for a given value of the Taylor number T, the length of the slug initially increased with N, but then approached a uniform length L, for values of N exceeding about 5. In fact Maxworthy found that L was almost the same for $N = 10$ and for $N = 100$. Maxworthy also determined the way in which L varied with the Taylor number and found that it increased almost linearly with T.

Maxworthy attempted to estimate the length of this column of trapped fluid by means of qualitative arguments based on the spread of the shear layers which separated the stagnant fluid from the main flow. He found that the observed length of the column was an order of magnitude smaller than that suggested by his arguments. He discussed this point in detail (see pp. 467–469) and concluded that his experiments were made under conditions which were "far from the slow-flow limit".

Symmetry properties of the wakes

A feature of the MS theory, valid for large values of $T^{\frac{1}{2}}$ and N, is that the velocity field ahead of and behind the body is symmetric (see also FM, § 6). In interpreting these theories it should be noted that they apply to the wakes generated by the body and that what happens very near the body is not really covered by the theories. Unfortunately it is not possible to ascertain the symmetry properties of the wakes from Maxworthy's (1970) experiments because no velocity measurements to the rear of the body are given. However, Maxworthy does give one photograph similar to that shown below in figure 2 (plate 1) (see his figure 13, plate 4) in which the forward and rear wakes are shown. The conditions for this experiment were $N = 7{\cdot}9$ and $T = 386$, and even though the value of N is rather small, the structures of the forward and rear wakes (away from the immediate neighbourhood of the body) do not appear to be greatly dissimilar: the forward wake is slightly broader than the rear wake and the magnitude of the swirl in the forward wake is less than that in the rear wake, but the similarities are such that it must remain an open question as to whether or not the wakes are nearly symmetric at large values of N and $T^{\frac{1}{2}}$.‡

† A correction for the effect of the finite thickness of the free shear layers in the theory of Moore & Saffman would almost certainly reduce the discrepancy between the theory and the measurements. Whether this would then lead to satisfactory agreement must await the appropriate calculations.

‡ Maxworthy gives a fairly comprehensive set of photographs relating to his dye studies. But it is virtually impossible to estimate from these the symmetry properties of the flow (unless the flow field is clearly asymmetric) because the coloured water influences the visualization in an asymmetric way. A comparison between figures 8 (*a*)–(*c*) and figure 13 of Maxworthy's paper indicates just how difficult it would be to estimate the symmetry properties of the wakes from the dye studies.

The flow near the body

In both his 1968 and his 1970 investigation Maxworthy reported a number of extremely interesting, small-scale features of the motions, especially in the region just to the rear of the sphere. It is possible that these features can have an important influence on the structure of the wakes, especially to the rear of the body, and accordingly we have looked closely to see if similar features were apparent in our stratified-fluid experiments.

Discussion

We found Maxworthy's experiments of considerable interest and help, first in planning our stratified-fluid experiment and later in interpreting the results. For us, the main points to arise from his results were as follows.

(i) *Measurements of the flow field in the wakes.* The most detailed and comprehensive information given by Maxworthy (1970) concerning the wakes consists of the measurements of the length of the column of fluid trapped ahead of the sphere. We are now able to give a fairly good theoretical prediction of the length of this column from the results of MS. To do this we define the length of the slug to be the largest distance from the body at which the fluid velocity on the axis ($\tilde{u}_0$) is equal to the speed of the body (U). The results of this calculation (which merely involves the evaluation of an integral from the MS results) are given below in table 2. Note that the MS theory is appropriate to the limit $T^{\frac{1}{2}} \rightarrow \infty$, $N \rightarrow \infty$ [statement (2.6)]. The interesting feature of the theoretical predictions is that the length of the column is independent of N and grows linearly with the Taylor number T. These predictions are in accordance with Maxworthy's results, namely that, for a given value of T, the length L was nearly constant for values of N exceeding about 10. Moreover, he found, for T in excess of about 200, that L increased almost linearly with T. The constant of proportionality is in very good agreement with the theoretical prediction (the difference between the empirical and the theoretical values being about 10%, as shown in table 2).

The length of the slug of blocked fluid ahead of the body is just one aspect of the entire flow generated by the body. Maxworthy has made some measurements of the velocity on the axis ahead of the forward slug, but the measurements were made only at small values of N and it is not reasonable to compare the MS theory with these results.

(ii) *The flow near the body.* Maxworthy's experiments yielded some unexpected results concerning the flow in the region immediately behind the sphere. These features could have an important influence on the structure of the wakes, especially the one to the rear of the body, but unfortunately it is not possible from Maxworthy's results to ascertain their effect on the wakes. The theoretical models of MS and FM are designed to predict the structure of the wakes. What happens near the body, or for that matter the detailed shape of the body, is not of great importance for the theories: the motion of the body merely supplies an axial (or horizontal, for the stratified-fluid case) velocity condition for the models. If that velocity condition is symmetric then the FM model for stratified flows necessarily

	Axisymmetric viscous rotating fluid (cf. MS theory)	Two-dimensional flows: Viscous rotating fluid ⇔ (cf. FM theory)	Viscous diffusive stratified fluid (cf. FM theory)	Viscous stratified fluids: Graebel's (1969) theory	Viscous stratified fluids: Janowitz's (1971) theory
Theoretical drag coefficient, C_D	$1{\cdot}71N$	$3{\cdot}14N$	$3{\cdot}14\,(\sigma Ri)^{\frac{1}{2}}$	$0{\cdot}667\,Ri$	∞
Experiment	$2{\cdot}60N$ (3)	None	None	None	
Theoretical length of column of trapped fluid, L/b	$5{\cdot}26\times10^{-2}\,T$	$7{\cdot}62\times10^{-2}\,T$	$3{\cdot}81\times10^{-2}\,Re(\sigma Ri)^{\frac{1}{2}}$	$9{\cdot}23\times10^{-3}\,Re\,Ri$	$1{\cdot}68\times10^{-2}\,Re\,Ri$
Experiment	$5{\cdot}88\times10^{-2}T$ (3)	None	See note (ii)	See note (iii)	
Theoretically symmetric?	Yes	Yes	Yes	No	No
Experiment	Not known	None	Apparently symmetric (1)	Apparently not symmetric (1), (2)	

Notes. (i) The theoretical values for L/b were determined as follows. (*a*) Rotating viscous fluid: by a fairly straightforward integration of the MS results (p. 630). (*b*) Viscous diffusive stratified fluid: by a calculation similar to that of (*a*) using the FM results. (*c*) Viscous stratified fluid (Graebel): according to Graebel the stream function ψ is given by

$$\psi(x,z) = zH\left(\frac{z+1}{X^{\frac14}}\right) - zH\left(\frac{z-1}{X^{\frac14}}\right) + 4X^{\frac14}\left[H'''\left(\frac{z-1}{X^{\frac14}}\right) - H'''\left(\frac{z+1}{X^{\frac14}}\right)\right], \tag{4.1}$$

where

$$X = x(Re\,Ri)^{-1}, \quad H(\eta) = \frac{1}{2} + \frac{1}{\pi}\int_0 t^{-1}\exp(-t^4)\sin\eta t\,dt.$$

Broward & Winant evaluated $H(\eta)$ numerically and then proceeded to compute ψ. However, Dr D. D. Freund has noticed that the velocity $u(x,z)$ can be computed by a more direct method. Since $H'''' = \frac14\eta H'$ it follows from (4.1) that

$$u = \psi_z = H\left(\frac{z+1}{X^{\frac14}}\right) - H\left(\frac{z-1}{X^{\frac14}}\right) - X^{-\frac14}H'\left(\frac{z+1}{X^{\frac14}}\right) - X^{-\frac14}H'\left(\frac{z-1}{X^{\frac14}}\right).$$

Thus

$$u(x,z) = \frac{2}{\pi}X^{-\frac14}\int_0^\infty \exp(-t^4)\cos(X^{-\frac14}t)\,[\sin(X^{-\frac14}t)/(X^{-\frac14}t) - \cos(X^{-\frac14}t)]\,dt.$$

(*d*) Viscous stratified fluid (Janowitz): the theoretical value is given in Janowitz's paper (p. 180).

(ii) For the conditions giving the best approximation to the FM limit we found experimentally (see figure 3(*d*) below) that $L/b \approx 21$ whereas FM gives $L/b = 35$.

(iii) For the present experiments (see figures 3(*d*) below) we found that $L/b \approx 14$ whereas Graebel's theory gives $L/b = 19$ and Janowitz's theory gives $L/b = 37$. Brownand & Winant's experiments yielded values for L/b in the range 19–30: Graebel's theory gives L/b in the range 22–220 and Janowitz's theory gives L/b in the range 39–390.

TABLE 2. A comparison between theoretical and experimental results for rotating and stratified flows. Numbers in parentheses: 1, present experiments; 2, Browand & Winant (1972); 3, Maxworthy (1970).

yields symmetric wakes fore and aft of the body (see appendix), and a similar result is likely to apply to the MS model. If, on the other hand, the velocity condition supplied by the body is not symmetric fore and aft there is no reason to suppose that the structures of the forward and rear wakes should be the same. This could have an important influence on the predicted properties such as the drag on the body.

(iii) *The drag.* Maxworthy's experiments were made with such great care and attention to detail that we feel that the discrepancy (of about 50%; see table 2) between the observed and theoretical drags requires explanation.

In his 1970 paper Maxworthy suggests that the values of N for the experiments were not nearly large enough for the asymptotic theories to be valid, although his drag measurements were made for values of N up to 200 (and the implications of figure 6 of Maxworthy's paper are that the differences would persist to very much larger values of N). However, the extremely good prediction from the MS theory of the length of the column of fluid trapped in front of the body for $N \approx 100$ [see (i) above] suggests to us that the values of N were large enough for the asymptotic theory to be a good candidate for modelling the experiments. If this is indeed the case, the poor theoretical prediction of the drag becomes a very important matter.

A detailed knowledge of the symmetry properties of the wakes would be of great benefit in assessing the usefulness of the MS model to describe Maxworthy's experiments. It is our guess that at the larger values of N in Maxworthy's experiments the velocities at comparable positions of the forward and rear wakes were not very different. If this was the case, then it would appear either that the MS model is inappropriate or that the drag is extremely sensitive to (what we suspect were only) very small differences in the structures of the wakes.

Our conclusion is that, although some of the measurements are described very well by the MS model, some are described very poorly, and thus it should be used only with great caution. Because of the similarities between viscous rotating fluids and viscous diffusive stratified fluids we think that similar caution should be exercised in applying the FM theory to experimental situations.

For convenience we have summarized, in table 2, the main theoretical predictions and experimental results for both rotating and stratified fluids.

5. Experimental apparatus

The experiments described in this paper were performed in a tank 3 m long with a 30×30 cm cross-section; the side walls were made from plate glass. Initially the body used to generate the motions was a piece of rectangular bar of cross-section $10 \times 9{\cdot}5$ mm, but some of the early results suggested that the horizontal boundary layers on the upper and lower surfaces of this body were influencing the motions slightly. Therefore a body of the shape shown in figure 1 was constructed from Perspex.

Near the upper and lower surfaces the body was made as thin as practicable so that there, at least, it would be a good approximation to a thin plate moving broadside through the fluid. Over the central portion it was made much thicker

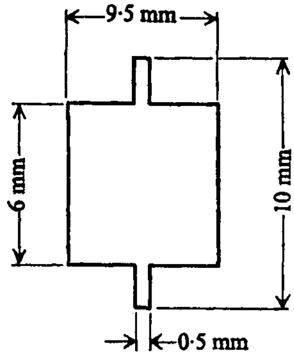

FIGURE 1. The shape of the body used for the experiments.

in order that it should be reasonably strong. The length of the body was made slightly smaller than the width of the channel and a horizontal slit was cut in each end of the body to a depth of about 10 mm. This slit was packed with a foam plastic and the same material was glued on to the ends of the body to ensure that there would be a good seal between the body and the walls of the tank. Two taut guide wires, which stretched between the ends of the tank, passed through the slits in the body. These were to prevent any vertical motion of the body as it moved along the tank. The body was towed along the tank by two other wires attached to the body near its ends. These wires stretched horizontally to one end of the tank, where they passed over pulleys to a winch driven by a speed-controlled motor. To maintain tension in the tow wires a similar pair stretched from the other side of the body to the end wall, where they passed over pulleys to a counterweight.

The tank was filled with a brine solution whose density increased linearly with height. The brine was introduced through the bottom of the tank and its concentration was controlled by an automatic system which increased the density of the inflow linearly with time. The filling time for the tank was about 4 h, but the contents were ready for use, with a good linear density gradient, shortly after the filling had been completed. Small samples were withdrawn from the tank to establish the actual density gradient in the tank. The scale height h of the density gradient was usually about 600 cm.

Two methods of visualizing the flow were used. In one of these a number of small crystals of potassium permanganate were dropped into the tank at convenient positions and the dye lines left by these crystals in falling to the bottom were photographed from time to time as the lines distorted with the flow. Examples of some of these photographs are shown in the plates.

The second method of flow visualization employed a schlieren system which uses the fact that the refractive index of salt solutions depends on the density. With this system a parallel beam of light enters the tank horizontally (normal to the side walls) and in passing through the solution in the tank each ray is refracted through an angle related to the local density gradient of the fluid. If the density gradient in the tank is everywhere uniform the light emerges as a parallel beam which may be brought to a sharp focus by a lens, and an image of the tank can be formed on a screen placed beyond this focus. If a razor edge is now inserted at the focal point of this beam to mask the rays, no light will reach the screen. But when disturbances are made to the density gradient in the tank these cause different rays to be refracted by different amounts, so that a sharp focus is no longer attained at the razor edge and some rays may pass the razor and form bright areas on the screen. Such a system can be made sensitive to very small perturbations of the density gradient, the sensitivity being determined by the size of the beam at the focal plane and the distance of this 'spot' from the razor edge. Variations of the system are possible and, for example, if the razor initially is brought very close to (but not quite to the point of masking) the beam at the focal point, the tank is imaged on the screen as a bright field and disturbances to the density gradient, of the appropriate sign, result in dark patches appearing on the screen. Moreover, should a slit be used instead of a single razor edge, a perturbation to the density gradient of either sign can result in dark patches appearing in the image of the tank; the magnitude of the density perturbations needed for the formation of these patches is related to the width of the slit at the focal plane of the light beam.

In the present experiments we have used this system for two purposes. First, we were able quickly to estimate whether or not the automatic filling device had worked properly (i.e. that the initial density gradient was nearly linear) from the uniformity of the image as the focal 'spot' was brought very close to the edge of the razor. This check was made before each experiment was begun.

Second, the system was used to check the flow field fore and aft of the body for symmetry. Because the schlieren system is sensitive to gradients of density, it can be quite difficult to make good quantitative interpretations that are easily checked against theories. However, if one is interested only in establishing whether or not the density perturbations fore and aft of the body are nearly symmetric, the schlieren system can give an immediate, overall view of this property.

Browand & Winant (1972) suggest, for their experiments, that the time scale of the transients which arose when the body was set in motion was about $20 \times (2b/U)$, and that this seemed to be roughly independent of the Richardson number and the Reynolds number. Thus, they suggest that, after the body had moved a distance of about 20 times its height, the fluid motions in the tank were nearly steady.† For the present experiments we were concerned that the transients should have nearly disappeared by the time the measurements of the flow field were begun, and although this was fairly difficult to estimate from the observations, our measurements indicated that the criterion suggested by Browand & Winant was probably a fairly reliable one to use. Therefore, to be safe, we usually started the body about 30 cm from the middle of the tank and moved it towards the further end. The observations of the flow field were usually made when the body was near the middle of the tank. In these experiments the speed of the body lay in the range 3×10^{-3} to 130×10^{-3} mm/s.

† There is, of course, an unsteady component of the flow occurring on a much larger time scale, which arises because the body moves towards one end of the tank and away from the other. Hopefully this effect is relatively unimportant when the body is near the centre of the tank.

The temperature of the solution was 19·5 °C for all the experiments. For salt solutions at this temperature, and with the range of speeds mentioned above, the Richardson number b^2g/U^2h varied between 5×10^6 and 2×10^3, the Reynolds number Ub/ν lay between 0·6 and 0·02, and the Schmidt number ν/ϑ was 1000.

6. Experimental results

In figure 2 (plate 1) are some examples of the way a vertical dye line distorts as a result of the motions generated by a body moving from left to right. The horizontal lines near the middle of the photograph are the guide wires along which the body moved. They are at the level $\tilde{z} = 0$. The quantity $\tilde{t}$ referred to in the caption is the period of time between the start of the motion and the instant at which the photograph was taken. The dye traces were formed at $\tilde{t} \approx 2500$ s. For the experiment shown in figure 2 the Richardson number was $2{\cdot}4 \times 10^3$, however the results are fairly typical of all the experiments carried out: most of the fluid motions were confined to a central region of about four or five body heights and outside this region the dye line was almost motionless. Thus, in figure 2 (*a*), we see that there was a wake extending well behind the body, and from figure 2 (*d*) it appears that the wake also extended well ahead of the body.

Because of the symmetry of the motions about the horizontal plane passing through the centre of the body (the plane $\tilde{z} = 0$), fluid particles at that level have no vertical velocity and therefore it is possible to measure velocities at $\tilde{z} = 0$ from a sequence of photographs of the kind shown in figure 2. Let this velocity be $\tilde{u}_0$, as in § 1. In figure 3 are shown the results of some of these measurements for various values of the body speed U; the graphs are arranged in order of decreasing U.† At the largest speed ($U = 126 \times 10^{-3}$ mm/s) the Richardson number was $2{\cdot}4 \times 10^3$ and the Reynolds number was 0·6, and we see from figure 3 (*a*) that, under these conditions, the velocities observed ahead of the body are quite different from those to the rear (at corresponding distances from the body). In particular, the rear wake appears to be quite feeble in comparison with the forward wake.

Similar results were obtained with $U = 92{\cdot}5 \times 10^{-3}$ mm/s, as shown in figure 3 (*b*). For this speed the limit (2.4) appeared to be best approached for the range of conditions under which the experiments were conducted (cf. table 1) and so the theoretical values of $\tilde{u}_0/U$ predicted by Graebel (1969) and by Janowitz (1971) are included. The predicted curve from Graebel's theory was calculated from the formula given in the footnote to table 2. It should, however, be noted that these curves relate only to the motions in front of the body; behind the body the velocity decays towards zero so rapidly with $\tilde{x}$ that the theoretical curves are virtually indistinguishable from the co-ordinate axes. In front of the body the agreement with Graebel's theory is fairly good (especially since the moderately short tank must have some effect on the results), whereas Janowitz's theory does

† The meaning to be attached to the error bars is discussed below.

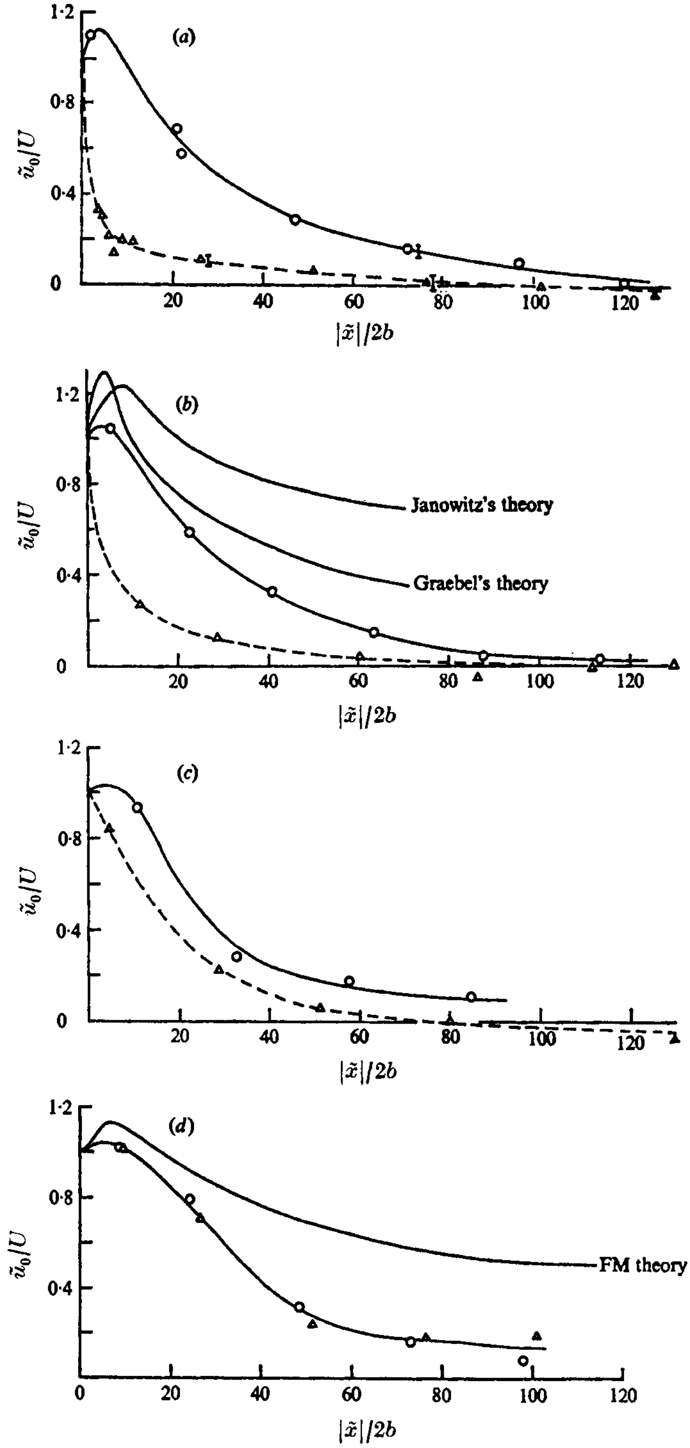

FIGURE 3. Measurements of the fluid velocity $\tilde{u}_0$ ahead of and behind the body for various values of the body speed. ○, $\tilde{x} > 0$; △, $\tilde{x} < 0$. (Note that $\beta = 1200$.) (*a*) $U = 126 \times 10^{-3}$ mm/s. (*b*) $U = 92{\cdot}5 \times 10^{-3}$ mm/s; the theoretical curves are for the motions ahead of the body. (*c*) $U = 39{\cdot}6 \times 10^{-3}$ mm/s. (*d*) $U = 3{\cdot}25 \times 10^{-3}$ mm/s; the theoretical curve is for the motions both ahead of and behind the body.

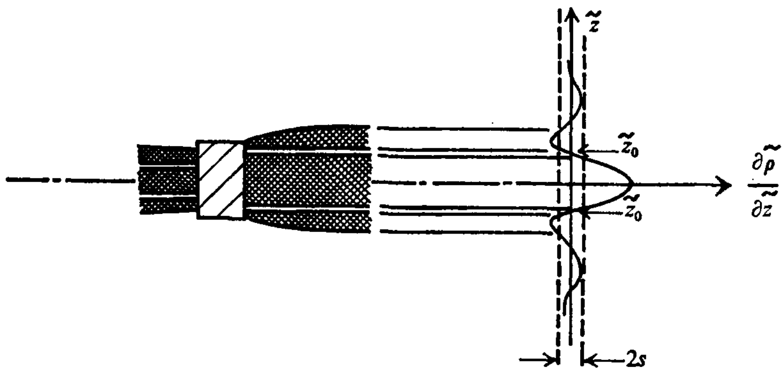

FIGURE 5. Sketch illustrating the interpretation of the schlieren image.

not appear to agree quite as well with the measurements.† To the rear of the body there was a distinct wake, in contrast to the predictions of both these theories, but this wake was considerably weaker than that ahead of the body and it seems to us very likely that a better approximation to the limit (2.4) would yield results even more closely in agreement with the viscous non-diffusive theories.

At smaller speeds, as for figure 3 (*c*), the wake to the rear of the body appeared to be stronger than in the previous two cases, and in figure 3 (*d*) the velocities to the rear of the body are nearly the same as those ahead. For the latter experiment the body speed was $3{\cdot}25 \times 10^{-3}$ mm/s and the Richardson number was $4{\cdot}5 \times 10^6$. Also shown in figure 3 (*d*) is the velocity distribution predicted by FM (this applies both ahead of and behind the body). The general agreement with the experimental results is fairly good if one considers that the ends of the tank must affect the motions. Certainly the results suggest that there is a tendency for the motions to become more and more nearly symmetric as the Richardson number increases. This tendency was also indicated by the schlieren observations.

Some photographs of schlieren images are shown in figure 4 (plate 2) for three different values of U. In each case the body is near the centre of the photograph and was moving from left to right. The vertical lines on the walls of the tank are 5·0 cm apart. These schlieren photographs were formed with a light field, as discussed in § 5. The beam was brought to a focus at the middle of a small slit between two horizontal razor edges so that the undisturbed image was a uniform bright field. Thus, perturbations to the density gradient could give rise to dark patches on the image, which appeared when $|\partial\tilde{\rho}/\partial\tilde{z}| > s$, where $\tilde{\rho}$ is the density perturbation and s is some number related to the sensitivity of the schlieren system (for example, s depends upon the width of the slit between the razor edges). A sketch is given in figure 5 of how to interpret the image.

Since the appearance of the dark patches depended on $|\partial\tilde{\rho}/\partial\tilde{z}|$ the schlieren

† In his paper Janowitz comments that it is necessary, for the validity of the theory, that $U^2h/gb^2 \ll Ub/\nu \ll 1$. For the present experiments, when $U = 100 \times 10^{-3}$ mm/s, $U^2h/gb^2 = 2 \times 10^{-4}$ and $Ub/\nu = 0{\cdot}5$; when $U = 1 \times 10^{-3}$ mm/s, $U^2h/gb^2 = 2 \times 10^{-8}$ and $Ub/\nu = 0{\cdot}005$. Thus it would appear that the lower the speed the better Janowitz's criterion is satisfied. However, FM point out that Janowitz's model is a limit solution of the exact equations only in the limit (2.4), and this appears to be best approached (in the present case) when $U \approx 100 \times 10^{-3}$ mm/s.

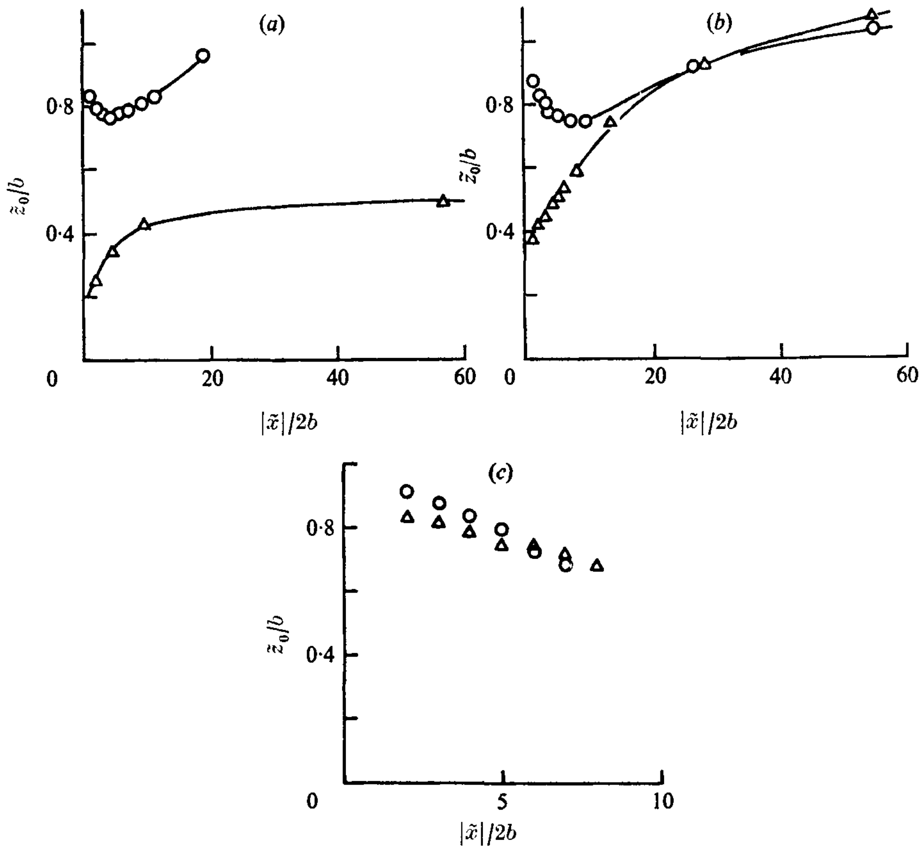

FIGURE 6. Measurements of $\tilde{z}_0/b$ from the schlieren photographs. $\beta = 1200$. ○, $\tilde{x} > 0$; △, $\tilde{x} < 0$. (a) $U = 127 \times 10^{-3}$ mm/s. (b) $U = 39{\cdot}5 \times 10^{-3}$ mm/s. (c) $U = 3{\cdot}54 \times 10^{-3}$ mm/s.

images should, according to the FM theory, have been symmetric in both $\tilde{x}$ and $\tilde{z}$. As a way of assessing the symmetry of photographs of the kind shown in figure 4 we have measured the distance $2\tilde{z}_0$, defined in figure 5, as a function of $\tilde{x}$. The results of these measurements are given in figure 6. As the photographs clearly show, the fore and aft wakes were not symmetric at the larger speeds, but we see from figure 6 (c) that at the smallest there was very little difference between the values of $\tilde{z}_0/b$ ahead of and behind the body. Unfortunately the body speed was so small in this experiment that the resulting perturbations to the density gradient were only just large enough to make our schlieren system workable, and this greatly restricted the distance over which we could measure $\tilde{z}_0$, as indicated in figure 6. In spite of this, the schlieren measurements appear to confirm the main conclusions drawn from the dye-trace measurements: at the larger body speeds there was a definite wake to the rear of the body, but this had quite different properties from the wake ahead of the body; as the body speed was reduced these differences became less marked, and at the smallest speed there appeared to be very little difference at all between the forward and aft wakes.

From the dye-line photographs we have also made some measurements of the horizontal velocity $\tilde{u}$ as a function of $\tilde{z}$, but in order to make the measurements

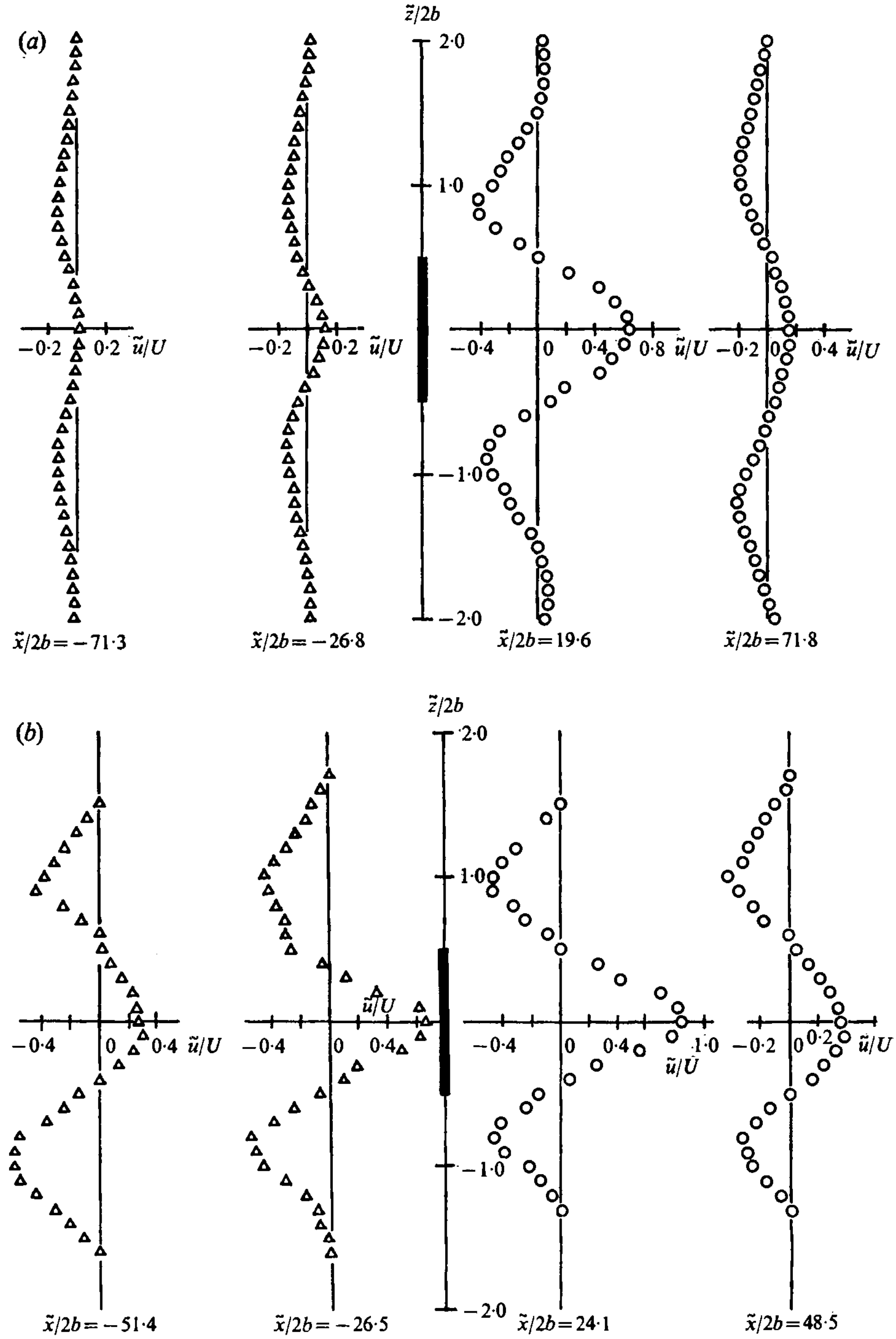

FIGURE 7. The horizontal velocity $\tilde{u}(\tilde{z})$ at various positions ahead of and behind the body. $\beta = 1200$. (a) $U = 126 \times 10^{-3}$ mm/s. (b) $U = 3·25 \times 10^{-3}$ mm/s.

we had to assume that the vertical velocities were negligible. The results of some of these measurements are shown in figures 7 (*a*) and (*b*), for which the experimental conditions were the same as those for figures (3 *a*) and (*d*) respectively. Again the results indicate that the structures of the wakes at the two body speeds were quite different. At the larger speed the wake to the rear was much more feeble than that ahead of the body, whereas at the smaller speed the fore and aft velocity distribution was more nearly symmetric. There is, however, an experimental difficulty associated with these measurements: if at a given value of $\tilde{x}$ an estimate of the mass flux is made by evaluating

$$\int_{-2\cdot 0}^{2\cdot 0} \tilde{u} d(\tilde{z}/2b),$$

the answer differs significantly from zero. (In fact the shape of the dye line in figure 2 (*a*) suggests that it would be difficult to balance the horizontal mass flux if the part of the dye line away from the central region did not move.) This effect was most noticeable near the ends of the tank when the body moved at its largest speed. Unfortunately we are not able to give a good explanation of this discrepancy, although we have made the following observations in connexion with it.

(i) At the larger values of $\tilde{x}$ the parts of the dye line well above and below the level $\tilde{z} = 0$ did not remain stationary but moved by a small amount. This 'drift' velocity appeared to be independent of $\tilde{z}$ and was in the wrong direction to account for the discrepancy.

(ii) In the experiments we were unable to prevent quite large disturbances to the fluid near the free surface. These disturbances were confined to a layer 2–3 cm thick and did not appear to interact with the motions generated by the movement of the body. This was supported by the observation that, well away from the level $\tilde{z} = 0$, there was no significant displacement of the dye lines during the experiment at the smallest body speed. Because we allowed a longer time to elapse between successive photographs at the smaller speeds the effect mentioned in (i) should have been more apparent at these speeds if it was a result of influences extraneous to the main experiment. Thus we suspect that the discrepancy in the mass-flux balance is associated with the motion of the body.

(iii) During the experiments we often viewed the dye traces from above the tank to check that the motions were nearly two-dimensional. Motions across the tank appeared to be relatively unimportant, however we must admit that we did not investigate this possibility very carefully near the ends of the tank.

Because we are unable to explain satisfactorily this discrepancy in the mass flux we have computed the (uniform) velocity that would have to be added to the measured velocities to ensure that the mass flux balances in the region $-2\cdot 0 \leqslant \tilde{z}/2b \leqslant 2\cdot 0$ of the tank. Twice the magnitude of this computed velocity is indicated by the vertical extent of the error bars shown in figure 3 (*a*). (Note that these bars are not intended to indicate our estimate of the absolute accuracy of the measurements.) At $\tilde{x}/2b \approx 20$ the bar would lie within the circle circumscribing the data point, if it had been drawn. At the smallest body speed (cf. figure 7 *b*) the discrepancy appeared to be much less significant and if included in figure 3 (*d*), as it was in figure 3 (*a*), the bar would lie within the circles and triangles circumscribing each of the data points.†

Thus the error associated with this discrepancy appears not to be too large, and although we remain very concerned about not being able to explain the nature of the inconsistency, we feel reasonably confident about our earlier interpretation of the results.

The flow field near the body

Maxworthy (1968, 1970) found in his experiments with rotating fluids that there was a considerable amount of small-scale structure to the motions, especially in the region just behind the body. He found wavelike disturbances in the shear layers behind the body and in the core of the wake immediately to the rear of the body. Therefore we have looked fairly carefully to see if the present experiments displayed similar features. We did not observe any of the wavelike motions that Maxworthy found in the rotating-fluid experiment, but the present results are probably best illustrated by the photographs in figure 8 (plate 3) and figures 9 and 10 (plate 4). These show how a vertical dye streak just in front of the body deforms as the body approaches it. In figure 8 (*a*), for which the body speed $U = 126 \times 10^{-3}$ mm/s, the dye trace has just begun to deform. Figures 8 (*b*)–(*d*) show how the dye particles in the layers just above and below the forward column of trapped fluid passed along the rear face of the body and almost coalesced at the level $\tilde{z} = 0$. An indication is given in figures 8 (*e*) and (*f*) of the form of the wake to the rear of the body. A similar sequence of photographs, for $U = 92{\cdot}9 \times 10^{-3}$ mm/s, is shown in figures 9 (*a*)–(*c*) (plate 4). At this speed there was a distance of about 1 mm between the two sets of dye particles as they moved away (almost horizontally) from the rear face of the body. A photograph of the same kind, for $U = 5{\cdot}73 \times 10^{-3}$ mm/s, is shown in figure 10 (plate 4). Although this photograph is not very distinct it has been included to show how, at this low speed, the dye particles from the main shear layers above and below the body did not move along the rear face of the body when they passed the edge of the body, but continued to move rearwards at almost the same horizontal level as that at which they passed round the body. (In moving past the edge of the body these dye particles changed their vertical level by about half a millimetre.)

Figure 11 (plate 5) shows a sequence of photographs taken during the early stages of an experiment for $U = 93{\cdot}7 \times 10^{-3}$ mm/s. The small mark on the wall of the tank, just below the horizontal wires, indicates the position of the rear face of the body before its motion was begun. Before the body was moved some small pellets of potassium permanganate, from which the dye traces were formed, were dropped onto the upper surfaces of the body. This formed a fairly intense cloud of dye which, shortly after the beginning of the motion, appeared (see figure 11 *a*) as dark patches ahead of and behind the body. The dyed region ahead of the body still has approximately the same configuration (relative to the body) in figure 11 (*c*) as it does in figure 11 (*a*); by contrast the dyed region to the rear of the body changed quite dramatically and the almost-horizontal dye lines gradually moved closer to each other near the rear face of the body.

† In this paper the circles and triangles are meant only to define the data points and not to give any indication of the errors involved in the experiment.

We are indebted to Professor R. E. Meyer for suggesting this experiment to us, and to Dr D. D. Freund for his help and advice during the course of the work. We are also extremely grateful to the Royal Aircraft Establishment (Farnborough) for the loan of the optical equipment used for the schlieren system.

Appendix.

Symmetry of solutions of the linear diffusive model

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We confirm here the result referred to in § 2 concerning the symmetry of solutions of the partial differential equation

$$\partial^6\psi/\partial z^6 + \partial^2\psi/\partial x^2 = 0, \quad (x,z)\in D, \tag{A 1}$$

where D is the open region $\mathbb{R}^2\backslash\{(0,z):|z|\leqslant\theta\}$. The equation is subject to the symmetric boundary condition‡

$$\psi(0^+,z) = \psi(0^-,z) \quad \text{for} \quad |z|\leqslant\theta. \tag{A 2}$$

A precise statement may be formulated as follows.

Theorem. *Any solution of* (A 1) *with* (A 2) *in the function class* $C_b(-\infty,\infty;L^2)$ *is necessarily even as a function of* x.

By $C_b(-\infty,\infty;L^2)$ is meant the class of functions $\phi(x,z)$ defined on $\mathbb{R}^2$ which, for each x, lie in the class $L^2(\mathbb{R})$ as a function of z and for which the mapping $x\rightarrow\phi(x,\cdot)$ is bounded and continuous from $\mathbb{R}$ into L^2. Of course the solutions of (A 1) are defined only on the domain D, but (A 2) allows the solution ψ to be extended to a function defined on $\mathbb{R}^2$. This function class contains the solution given by Freund & Meyer (1972), namely

$$\psi_0(x,z) = \theta\int_0^\infty t^{-1}J_1(\theta t)\exp(-t^3|x|)\sin zt\,dt, \tag{A 3}$$

where J_1 is the first-order Bessel function of the first kind. Extended to $\mathbb{R}^2$ as above, ψ_0 has jump discontinuities at the ends $(0,\theta)$ and $(0,-\theta)$ of the body, and is $O(z^{-1})$ as $|z|\rightarrow\infty$, for each fixed x. These constraints lead naturally to the choice of the function class.

Proof. To prove this theorem we first consider a characteristic initial-value problem corresponding to (A 1) with (A 2), namely

$$\partial^2\phi/\partial x^2 + \partial^6\phi/\partial z^6 = 0 \quad \text{for} \quad (x,z)\in\mathbb{R}^+\times\mathbb{R}, \tag{A 4}$$

with

$$\phi(0,z) \equiv A(z)\in L^2(\mathbb{R}). \tag{A 5}$$

For this problem we have the following lemma.

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‡ For the problem studied by Freund & Meyer (1972), (A 2) takes the particular form

$$\psi(0^+,z) = \psi(0^-,z) = z \quad \text{for} \quad |z|\leqslant\theta.$$

LEMMA. *There is at most one solution of* (A 4) *with* (A 5) *in the function class* $C_b(0,\infty;L^2)$.

Given the validity of the lemma, the theorem follows trivially. For if ψ is any solution of (A 1) with (A 2) in $C_b(-\infty,\infty;L^2)$, let $\phi_1(x,z) = \psi(x,z)$ and $\phi_2(x,z) = \psi(-x,z)$ for $(x,z)\in\mathbb{R}^+\times\mathbb{R}$. Then the $C_b(0,\infty;L^2)$ functions ϕ_1 and ϕ_2 both satisfy (A 4) with the same boundary conditions (A 5) at $x = 0$, and therefore must agree everywhere in $\mathbb{R}^+\times\mathbb{R}$.

Proof of lemma. The function class is defined analogously to the class $C_b(-\infty,\infty;L^2)$, with the mapping $x\to\phi(x,\cdot)$ now being continuous and bounded from $[0,\infty)\to L^2(\mathbb{R})$. Let $\hat{\phi} = \hat{\phi}(x,s)$ denote the Fourier transform of ϕ in the z variable, where s is the transformed variable. Then, in the sense of distributions at least,

$$\partial_x^2\hat{\phi} = s^6\hat{\phi}. \tag{A 6}$$

From the continuity of the Fourier transform as an operator on L^2 we have that $\hat{\phi}\in C_b(0,\infty;L^2)\subset L^2(0,X;L^2) = L^2([0,X]\times\mathbb{R})$, for any finite $X > 0$, and the inclusion is a continuous injection. In particular, by Fubini's theorem, $\hat{\phi}(x,s)$ is in $L^2(0,X)$ for almost every s, and hence inductively from (A 6), $\hat{\phi}(x,s)$ is $C^\infty(0,X)$ for all finite $X > 0$ and for almost every s. Thus, for almost all s,

$$\hat{\phi}(x,s) = C_1(s)\exp(|s|^3x) + C(s)\exp(-|s|^3x) \tag{A 7}$$

holds for all $x \geqslant 0$. By hypothesis and an application of Plancherel's theorem

$$\|\hat{\phi}(x,\cdot)\|_{L^2} = \|\phi(x,\cdot)\|_{L^2} \leqslant M,$$

a result which holds uniformly for $0 \leqslant x < \infty$. From the representation (A 7), however, this can occur only if $C_1(s) = 0$, almost everywhere. Thus, for almost all s,

$$\hat{\phi}(x,s) = C(s)\exp(-|s|^3x)$$

holds everywhere in x, for such s. Since $\hat{\phi}\in C_b(0,\infty;L^2)$, the initial value $\hat{\phi}(0,s)$ is assumed in the L^2 sense so that, for almost all s,

$$C(s) = \lim_{x\downarrow 0}\hat{\phi}(x,s) = \hat{\phi}(0,s) = \mathscr{F}_z\phi(0,z) = \mathscr{F}_zA(z),$$

where $\mathscr{F}_z$ denotes the Fourier transform applied to the z variable. Thus $\hat{\phi}$ is determined uniquely within the class $C_b(0,\infty;L^2)$ by $A(z)$, and taking the inverse Fourier transform shows that $\phi(x,z)$ is uniquely determined in the same class by $A(z)$, as required.

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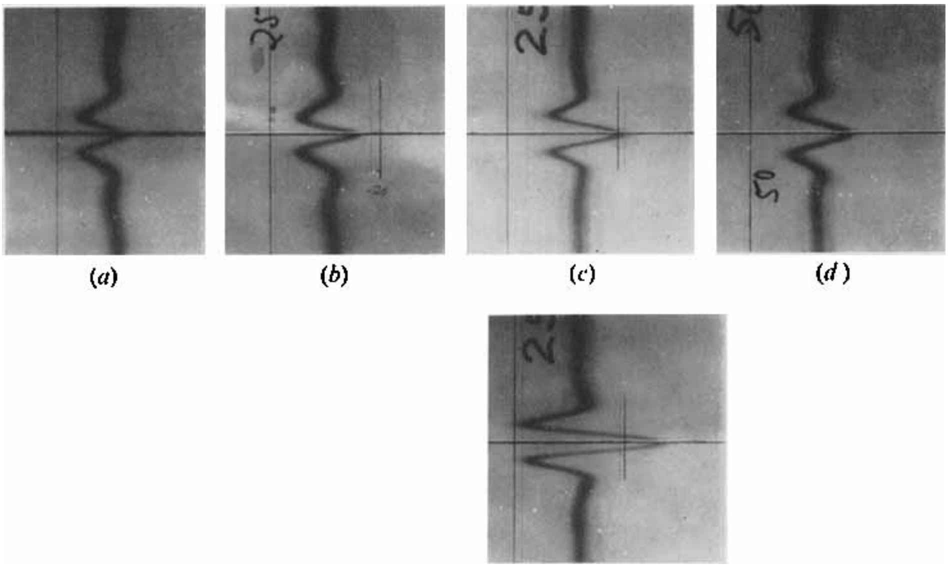
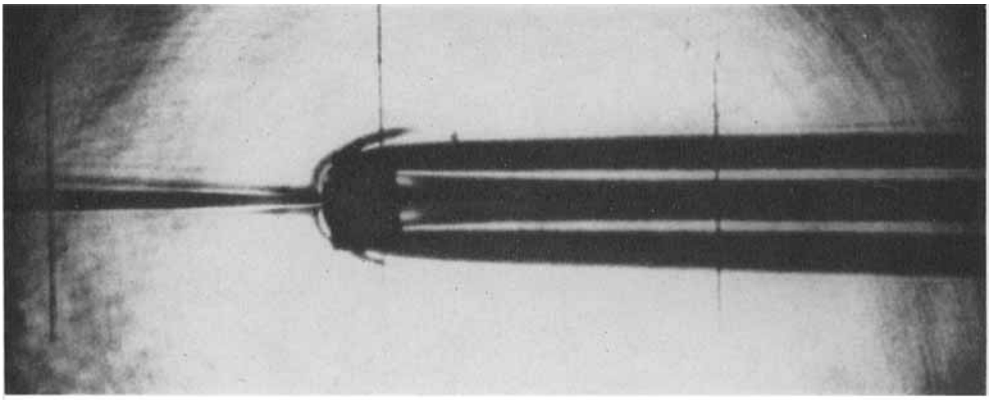
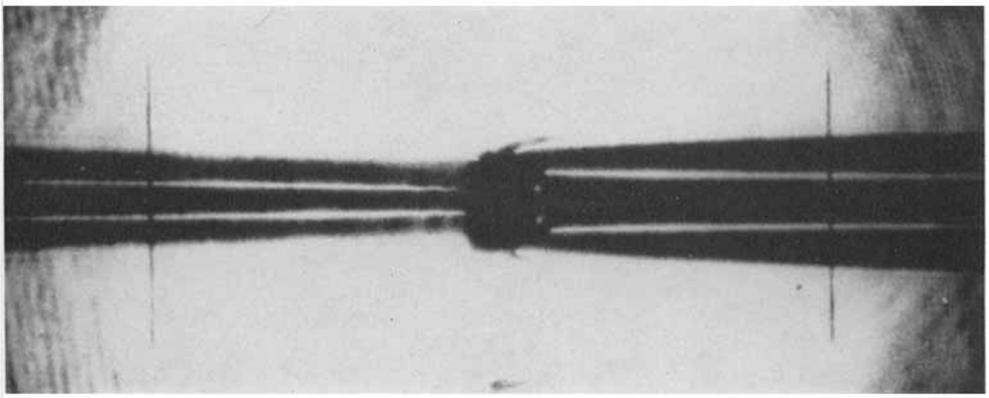


(e)

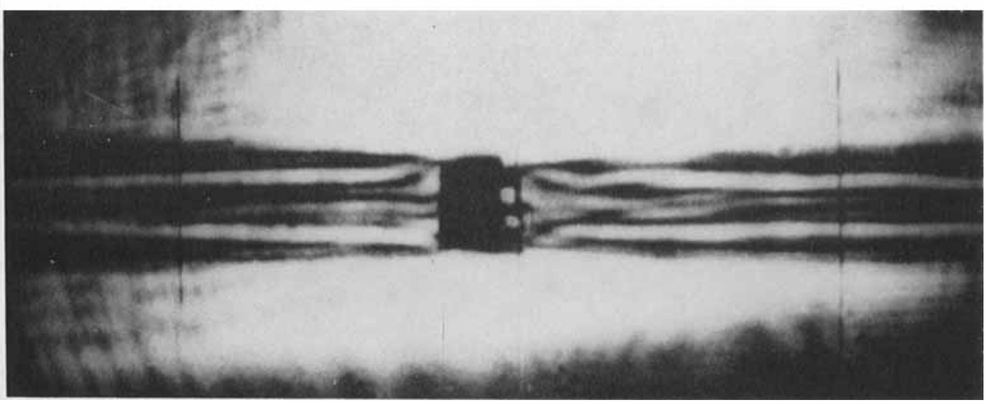
FIGURE 2. Photographs showing the distortion of dye traces for $U = 126 \times 10^{-3}$ mm/s, $\beta = 1200$. (*a*) $\tilde{t} = 3943$ s; the dye line is 62 cm behind the body. (*b*) $\tilde{t} = 3910$ s; the dye line is 32 cm behind the body. (*c*) $\tilde{t} = 3802$ s; the dye line is 19 cm ahead of the body. (*d*) $\tilde{t} = 3815$ s; the dye line is 44 cm ahead of the body. (*e*) The same dye trace as in (*c*) at $\tilde{t} = 4058$ s; in the time between these two photographs the body moved a distance of 3·23 cm. (The vertical lines in (*b*), (*c*) and (*e*) are on the walls of the tank. One set of these lines is on the side of the tank nearer the camera and the other set is on the side further away. The lines on a given wall are separated by 5·0 cm.)



(*a*)



(*b*)



(*c*)

FIGURE 4. Photographs of the schlieren images. $\beta = 1200$. (*a*) $U = 127 \times 10^{-3}$ mm/s. (*b*) $U = 39{\cdot}5 \times 10^{-3}$ mm/s. (*c*) $U = 3{\cdot}54 \times 10^{-3}$ mm/s.

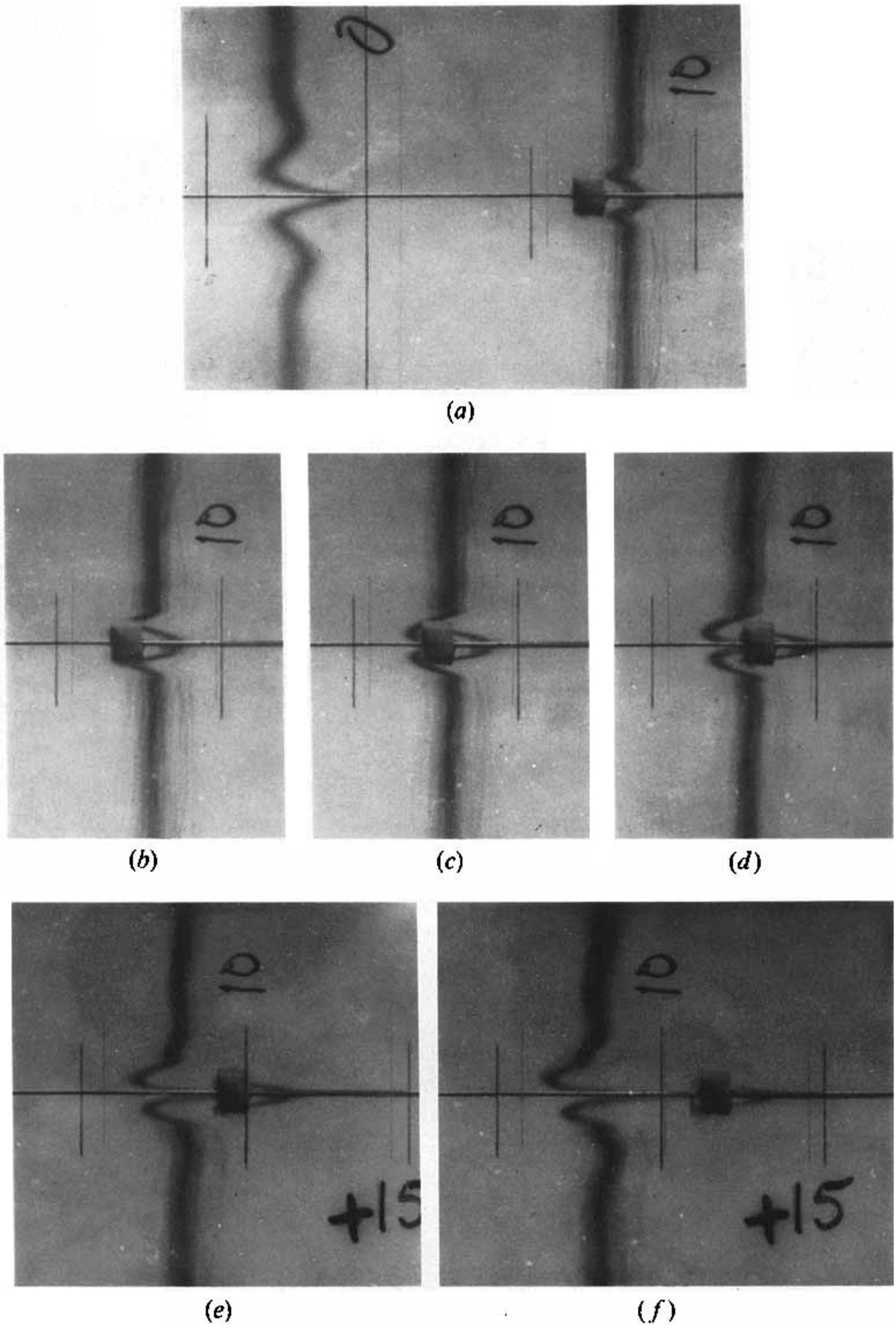
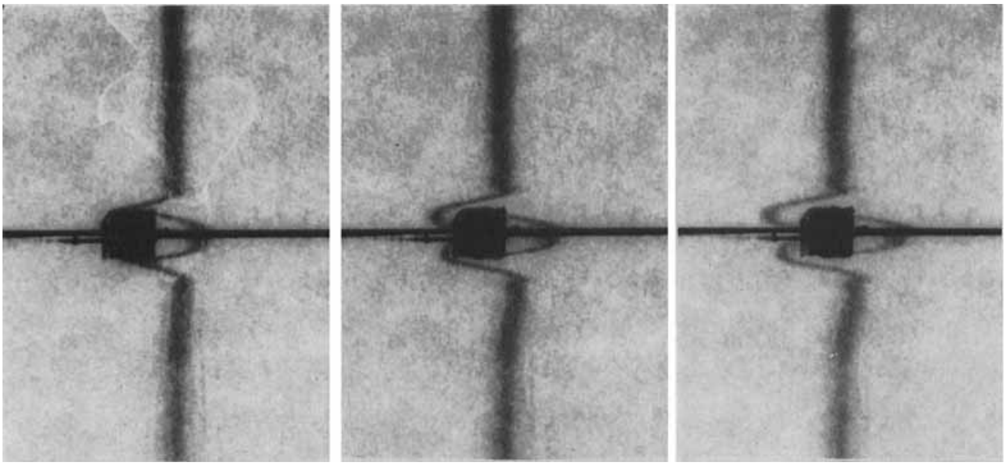


FIGURE 8. A sequence of photographs showing the way a vertical dye line near the body distorts for $U = 126 \times 10^{-3}$ mm/s, $\beta = 1200$. (*a*) $\tilde{t} = 3660$ s. (*b*) $\tilde{t} = 3690$ s. (*c*) $\tilde{t} = 3704$ s. (*d*) $\tilde{t} = 3778$ s. (*e*) $\tilde{t} = 3882$ s. (*f*) $\tilde{t} = 4443$ s.

BARNARD AND PRITCHARD



(*a*) (*b*) (*c*)

FIGURE 9. A sequence of photographs showing the way a vertical line near the body distorts for $U = 92{\cdot}9 \times 10^{-3}$ mm/s, $\beta = 1200$. (*a*) $\tilde{t} = 8043$ s. (*b*) $\tilde{t} = 8114$ s. (*c*) $\tilde{t} = 8150$ s.

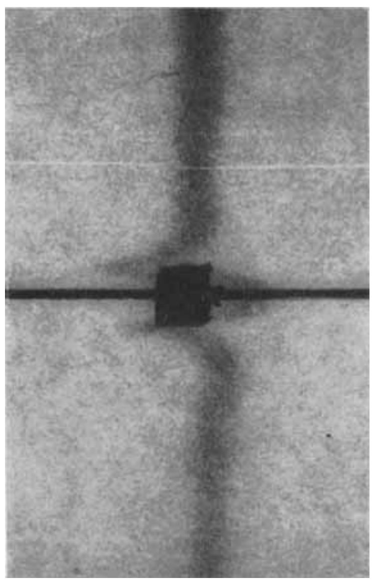
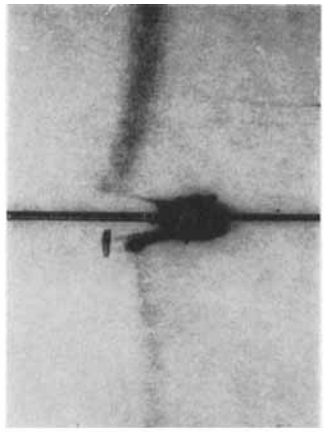
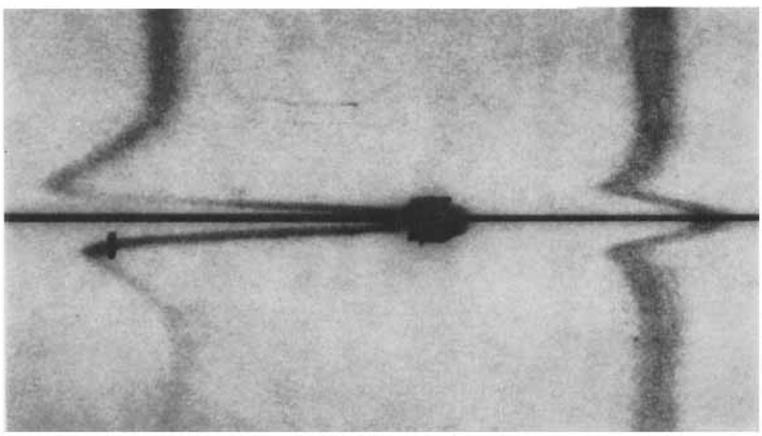


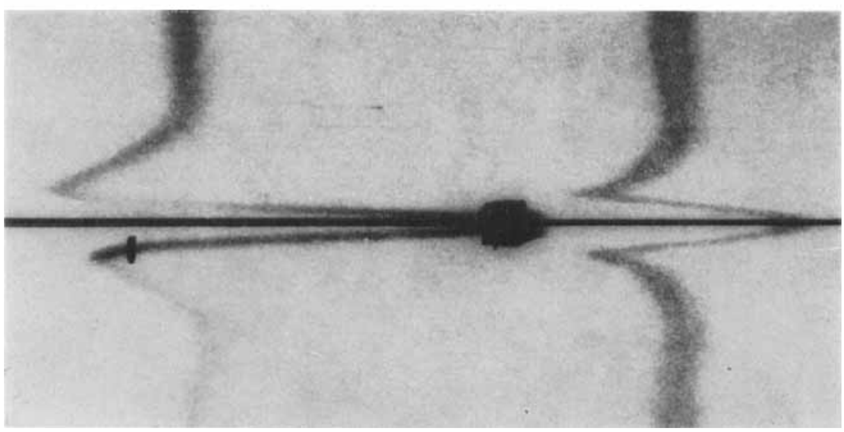
FIGURE 10. A photograph showing how a vertical dye line near the body distorts for $U = 5{\cdot}73 \times 10^{-3}$ mm/s, $\beta = 1200$.



(*a*)



(*b*)



(*c*)

FIGURE 11. A sequence of photographs showing how a dye line distorts during the initial stages of the motion for $U = 93{\cdot}7\times10^{-3}$ mm/s, $\beta = 1200$. (*a*) $\tilde{t} = 225$ s. (*b*) $\tilde{t} = 905$ s. (*c*) $\tilde{t} = 1085$ s. The small vertical line just below the horizontal wires indicates the position of the rear face of the body for $\tilde{t} \leqslant 0$.